The Renormalization of the Non-Abelian Gauge Theories in the Causal Approach

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Abstract

We consider the gauge invariance of the standard Yang-Mills model in the framework of the causal approach of Epstein-Glaser and Scharf and determine the generic form of the anomalies. The method used is based Epstein-Glaser approach to renormalization theory. In the case of quantum electrodynamics we obtain quite easily the absence of anomalies in all orders.

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1 Introduction

The causal approach to renormalization theory of by Epstein and Glaser [30], [31] had produced important simplification of the renormalization theory at the purely conceptual level as well as to the computational aspects. This approach works for quantum electrodynamics [52], [21], [36] where it brings important simplifications of the renormalizability proof. For Yang-Mills theories [23], [24], [26], [27], [3], [5], [17], [18], [40]-[43], [45], [28], [33], [34], [35], [53] one can determine severe constraints on the interaction Lagrangian (or in the language of the renormalization theory - on the first order chronological product) from the condition of gauge invariance. Gravitation can be also analysed in this framework [32], [38], [39], [58], etc. Finally, the analysis of scale invariance can be done [37], [50]. One should stress the fact that the Epstein-Glaser analysis uses exclusively the Bogoliubov axioms of renormalization theory [12] imposed on the scattering matrix: this is an operator acting in the Hilbert space of the model, usually generated from the vacuum by the quantum fields corresponding to the particles of the model. If one considers the S-matrix as a perturbative expansion in the coupling constant of the theory, one can translate these axioms on the chronological products. Epstein-Glaser approach is a inductive procedure to construct the chronological products in higher orders starting from the first-order of the perturbation theory. For gauge theories one can construct a non-trivial interaction only if one considers a larger Hilbert space generated by the fields associated with the particles of the model and the ghost fields. The condition of gauge invariance becomes in this framework the condition of factorization of the S-matrix to the physical Hilbert space in the adiabatic limit. To avoid infra-red problems one works with a formulation of this factorization condition which corresponds to a formal adiabatic limit and it is perfectly rigorously defined [24]. The obstructions to the implementation of the condition of gauge invariance are called *anomalies*. The most famous is the Adler-Bell-Bardeen-Jackiw anomaly [1], [6], [10], [7] (see [49] for a review).

The classical analysis of the renormalizablity of Yang-Mills theories of Becchi, Rouet, Stora and Tyutin [11] is based on a different combinatorial idea. Namely, one considers a perturbative expansion in Planck constant $\hbar$ which is equivalent, in Feynman graphs terminology, to a loop expansion. (The rigorous connection between these two perturbation schemes has been recently under investigation [22].) One can formulate the condition of gauge invariance in terms of the generating functional for the one-particle irreducible Feynman amplitudes; the S-matrix is then recovered using the reduction formulæ [51]. Presumably, both formulations lead to the same S-matrix, up to finite renormalization, although this point is not firmly established in the literature. The most difficult part is to prove that if there are no anomalies in lower orders of perturbation theory, then the anomalies are absent in higher orders. The main tool of the proof is the consideration of the scale invariance properties of a quantum theory expressed in the form of Callan-Symanzik equations [13], [14] and [15]. A mathematical analysis was developed in [56] and [57], using the quantum action principle [47] (for a review see [51]). One should stress the fact that in this approach one works with interaction fields which can be defined as formal series in the coupling constant. The main observation used in these references is the existence of anomalous dimensional behaviour of the (interacting) fields with respect to dilations. Based on this analysis in [9] (see also [11] and [51]) it is showed that the ABBJ anomaly can appear

only in the order $n = 3$ of the perturbation theory. A analysis of the standard model based on this approach can be found in [46].

In [37] we have investigated scale anomaly from the point of view of Epstein-Glaser causal approach based entirely on a perturbation scheme of Bogoliubov based on an expansion in the coupling constant. We have found out the surprising result that scale invariance does not restrict the presence of the anomalies in higher orders of perturbation theory. So, from the point of view of Bogoliubov axioms, the elimination of anomalies in higher orders of perturbation theory is still an open question.

The purpose of this paper is to investigate the generic form of the anomalies compatible with the restrictions following from covariance properties and formal gauge invariance. Our strategy will be based exclusively on the Epstein-Glaser construction of the chronological products for the free fields. The rôle of Feynman graph combinatorics is completely eliminated in this analysis. We will use in fact a reformulation of the Epstein-Glaser formalism [21] which gives a prescription for the construction of the chronological products of the type $T(A_1(x_1), \cdots, A_n(x_n))$ for any Wick polynomials $A_1(x_1), \cdots, A_n(x_n)$. The main point is to formulate a proper induction hypothesis for the expression $d_Q T(A_1(x_1), \cdots, A_n(x_n))$ where d_Q is the BRST operator[2]. In fact, we will see that it is necessary to make such a conjecture only for some special cases of Wick polynomials. If $T(x)$ is the interaction Lagrangian (i.e. the first order chronological product) one can prove the validity of some "descent" equations of the type:

$$d_Q T(x) = i\partial_\mu T^\mu(x), \quad d_Q T^\mu(x) = i\partial_\nu T^{\mu\nu}(x), \quad ,\ldots, \quad d_Q T^{\mu_1,\ldots,\mu_{p-1}}(x) = i\partial_{\mu_p} T^{\mu_1,\ldots,\mu_p}(x). \tag{1.0.1}$$

In the QED the procedure stops after the first step ($p = 1$) and in the Yang-Mills case after a two steps ($p = 2$). In general, one can consider the case when the descent stops after a finite number of steps. In this case one has to give a proper conjecture for $d_Q T(A_1(x_1), \cdots, A_n(x_n))$ only for $A_1(x_1), \cdots, A_n(x_n)$ of the type $T(x), \quad T^\mu(x), \quad T^{\mu\nu}(x), \quad \ldots.$

The structure of this paper is the following one. In the next Section we make a brief review of essential points concerning Epstein-Glaser resolution scheme of Bogoliubov axioms and the standard model in the framework of the causal approach. (For more details see [36] and [34]). We emphasize that the main problem is to establish the factorization of the S-matrix to the physical Hilbert space; in the formal adiabatic limit, this is the famous condition of gauge invariance. Translated in terms of Feynman amplitudes this condition amounts, essentially, to the so-called Ward-Takahashi identities, or - in the language of the Zürich group - the C-g identities. In the next Section we give the inductive hypothesis for quantum electrodynamics and prove that there are no anomalies. Next, we do the same thing for Yang-Mills theories and determine the generic form of the anomalies.

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2 Perturbation Theory in the Causal Approach

2.1 Bogoliubov Axioms

We present the main ideas of perturbation theory following [30] and [36] to which we refer for more details. The S-matrix is formal series of operator valued distributions:

$$S(\mathbf{g}) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{\mathbb{R}^{4n}} dx_1 \cdots dx_n \, T_{j_1,\dots,j_n}(x_1, \cdots, x_n) g_{j_1}(x_1) \cdots g_{j_n}(x_n), \tag{2.1.1}$$

where $\mathbf{g} = (g_j(x))_{j=1,\dots P}$ is a vector-valued tempered test function and $T_{j_1,\dots,j_n}(x_1, \cdots, x_n)$ are operator-valued distributions acting in the Fock space of some collection of free fields with a common dense domain of definition D_0. The scalar product is denoted by $(\cdot,\cdot)$. These operator-valued distributions are called *chronological products* and verify *Bogoliubov axioms*. One starts from a set of *interaction Lagrangians* $T_j(x), \quad j = 1, \dots, P$ and tries to construct the whole series $T_{j_1,\dots,j_n}, \quad n \geq 2$.

Usually, the interactions Lagrangians are Wick monomials. The canonical dimension $\omega(W)$ of certain Wick monomial is defined according to the usual prescription. By definition, a *Wick polynomial* is a sum of Wick monomials.

Bogoliubov axioms are quite natural and they describe the behaviour of the chronological products with respect to: (a) the permutation of the couples (x_i, j_i) (symmetry); (b) the action of the Poincaré group in the Fock space of the system (Poincaré invariance); (c) the factorization property for temporally successive arguments (causality); (d) the Hermitian conjugation (unitarity). We mention the essential rôle of causality in all approaches to quantum field theory [54], [55].

One considers a *interaction Lagrangian* given by a formula of the type:

$$T(x) = \sum c_j T_j(x) \tag{2.1.2}$$

with c_j some real constants. In this case, the chronological products of the theory are

$$T(X) = \sum c_{j_1} \dots c_{j_n} T_{j_1,\dots,j_n}(X) \tag{2.1.3}$$

and they must be plugged into an expression of the type (2.1.1) for $P = 1$ to generate the S-matrix of the model.

It can be showed that that one must consider the given interaction Lagrangians $T_j(x)$ to be all Wick monomials canonical dimension $\omega_j \leq 4$ $(j = 1, \dots, P)$ acting in the Fock space of the system.

If there are non-Hermitian free fields acting in the Fock space, we have in general:

$$T_j(x)^\dagger = T_{j^*}(x) \tag{2.1.4}$$

where $j \to j^*$ is a bijective map of the numbers $1, 2, \dots, P$.

If there are Fermi or ghost fields acting in the Fock space, the causality property is in general:

$$T_{j_1}(x_1) T_{j_2}(x_2) = (-1)^{\sigma_{j_1}\sigma_{j_2}} T_{j_2}(x_2) T_{j_1}(x_1), \quad \forall x_1 \sim x_2. \tag{2.1.5}$$

Here σ_i is the number of Fermi and ghost fields factors in the Wick monomial T_j; if σ_j is even (odd) we call the index j *even* (resp. *odd*). One has to keep track of these signs in the symmetry axiom for the chronological products.

It is convenient to let the index j have the value 0 also and we put by definition

$$T_0 \equiv \mathbf{1}. \tag{2.1.6}$$

Moreover, we define a new sum operation of two indices $j_1, j_2 = 1, \dots, P$; this summation is denoted by $+$ but should not be confused with the ordinary sum. By definition we have:

$$T_{j_1+j_2}(x) = c :T_{j_1}(x) T_{j_2}(x): \tag{2.1.7}$$

for some positive constant c. We define componentwise the summation for n-tuples $J = \{j_1, \dots, j_n\}$. The new summation is non-commutative if Fermi or ghost fields are present.

We will use the notation

$$\omega_J \equiv \sum_{j \in J} \omega_j \tag{2.1.8}$$

and we call it the *canonical dimension* of $T_J(X)$.

According to Epstein-Glaser [30] one must add a new axiom, namely the following *Wick expansion* of the chronological products is valid:

$$T_J(X) = \sum_{K+L=J} \epsilon \quad t_K(X) \quad W_L(X) \tag{2.1.9}$$

where: (a) $t_K(X)$ are numerical distributions (the *renormalized Feynman amplitudes*); (b) the degree of singularity is restricted by the following relation:

$$\omega(t_K) \leq \omega_K - 4(n-1); \tag{2.1.10}$$

(c) ϵ is the sign coming from permutation of Fermi fields; (d) we have introduced the notation

$$W_J(X) \equiv :T_{j_1}(x_1) \cdots T_{j_n}(x_n): \tag{2.1.11}$$

Let us notice that from (2.1.9) we have:

$$t_J(X) = (\Omega, T_J(X)\Omega) \tag{2.1.12}$$

where Ω is the vacuum state of the Fock space.

We end this Subsection with an important remark. Let us consider some general Wick polynomials

$$A_i(x) = \sum_j c_{ij} \quad T_j(x), \quad i = 1, 2, \dots \tag{2.1.13}$$

Then we can define the chronological products:

$$T(A_1(x_1), \cdots, A_n(x_n)) \equiv \sum_J c_{i_1 j_1} \cdots c_{i_n j_n} \quad T_{j_1,\dots,j_n}(x_1, \cdots, x_n). \tag{2.1.14}$$

One can find in [21] a system of axioms for the expressions $T(A_1(x_1), \cdots, A_n(x_n))$ which is equivalent to the Bogoliubov set of axioms.

2.2 Massive Yang-Mills Fields

In [33] - [35] we have justified the following scheme for the standard model (SM): we consider the auxiliary Hilbert space $\mathcal{H}^{gh,r}_{YM}$ generated from the vacuum Ω by applying the free fields $A_{a\mu},\ u_a,\ \tilde{u}_a,\ \Phi_a \quad a = 1,\dots,r$ where the first one has vector transformation properties with respect to the Poincaré group and the others are scalars. In other words, every vector field has three scalar partners. Also $u_a,\ \tilde{u}_a \quad a = 1,\dots,r$ are Fermion and $A_\mu,\ \Phi_a \quad a = 1,\dots,r$ are Boson fields.

We have two distinct possibilities for distinct indices a:

(I) Fields of type I correspond to an index a such that the vector field A^μ_a has non-zero mass m_a. In this case we suppose that all the other scalar partners fields $u_a,\ \tilde{u}_a,\ \Phi_a$ have the same mass m_a.

(II) Fields of type II correspond to an index a such that the vector field A^μ_a has zero mass. In this case we suppose that the scalar partners fields $u_a,\ \tilde{u}_a$ also have the zero mass but the scalar field Φ_a can have a non-zero mass: $m^H_a \geq 0$. It is convenient to use the compact notation

$$m^*_a \equiv \begin{cases} m_a & \text{for} \quad m_a \neq 0 \\ m^H_a & \text{for} \quad m_a = 0 \end{cases} \tag{2.2.1}$$

Then the following following equations of motion describe the preceding construction:

$$(\square + m_a^2) u_a(x) = 0, \quad (\square + m_a^2)\tilde{u}_a(x) = 0, \quad (\square + (m^*_a)^2)\Phi_a(x) = 0, \quad a = 1,\dots,r. \tag{2.2.2}$$

We also postulate the following canonical (anti)commutation relations:

$$\begin{gathered} [A_{a\mu}(x), A_{b\nu}(y)] = -\delta_{ab} g_{\mu\nu} D_{m_a}(x-y) \times \mathbf{1}, \\ \{u_a(x), \tilde{u}_b(y)\} = \delta_{ab} D_{m_a}(x-y) \times \mathbf{1}, \quad [\Phi_a(x), \Phi_b(y)] = \delta_{ab} D_{m^*_a}(x-y) \times \mathbf{1}; \end{gathered} \tag{2.2.3}$$

all other (anti)commutators are null.

In this Hilbert space we suppose given a sesquilinear form $< \cdot, \cdot >$ such that:

$$A_{a\mu}(x)^\dagger = A_{a\mu}(x), \quad u_a(x)^\dagger = u_a(x), \quad \tilde{u}_a(x)^\dagger = -\tilde{u}_a(x), \quad \Phi_a(x)^\dagger = \Phi_a(x). \tag{2.2.4}$$

The ghost degree is ± 1 for the fields u_a (resp. $\tilde{u}_a$), $\quad a = 1,\dots,r$ and 0 for the other fields.

One can define the BRST *supercharge* Q by:

$$\begin{gathered} \{Q, u_a\} = 0 \quad \{Q, \tilde{u}_a\} = -i(\partial_\mu A^\mu_a + m_a \Phi_a) \\ [Q, A^\mu_a] = i\partial^\mu u_a \quad [Q, \Phi_a] = i m_a u_a, \quad \forall a = 1,\dots,r \end{gathered} \tag{2.2.5}$$

and

$$Q\Omega = 0. \tag{2.2.6}$$

Then one can justify that the **physical** Hilbert space of the Yang-Mills system is a factor space

$$\mathcal{H}^r_{YM} \equiv \mathcal{H} \equiv Ker(Q)/Ran(Q). \tag{2.2.7}$$

The sesquilinear form $< \cdot, \cdot >$ induces a *bona fide* scalar product on the Hilbert factor space.

The factorization process leads to the following **physical** particle content of this model:

- For $m_a > 0$ the fields $A^\mu_a,\ u_a,\ \tilde{u}_a,\ \Phi_a$ describe a particle of mass $m_a > 0$ and spin 1; this are the so-called *heavy Bosons* [34].
- For $m_a = 0$ the fields $A^\mu_a,\ u_a,\ \tilde{u}_a$ describe a particle of mass 0 and helicity 1; the typical example is the *photon* [33].
- For $m_a = 0$ the fields Φ_a describe a scalar fields of mass m^H_a; this are the so-called *Higgs fields.*

This framework is sufficient for the study of the Standard Model (SM) of the electro-weak interactions. To include also quantum chromodynamics one must consider that there is a third case:

(III) Fields of type III correspond to an index a such that the vector field A^μ_a has zero mass, the scalar partners $u_a,\ \tilde{u}_a$ also have zero mass but the scalar field Φ_a is absent.

In [53] and [29] the model is constructed somewhat differently: one eliminates the fields of type II and includes a number of supplementary scalar Bosonic fields φ_i of masses $m_i \geq 0$. In this framework one can consider for instance the very interesting Higgs-Kibble model in which there are no zero-mass particle, so the adiabatic limit probably exists.

We can preserve the general framework with only two types of indices if we consider that in case II there are in fact three subcases (i.e three types of indices a for which $m_a = 0$):

(IIa) In this case $A_{a\mu},\ u_a,\ \tilde{u}_a,\ \Phi_a \not\equiv 0$;

(IIb) In this case $\Phi_a \equiv 0$;

(IIc) In this case $A_{a\mu},\ u_a,\ \tilde{u}_a \equiv 0$.

One must modify appropriately the canonical (anti)commutation relations (2.2.3) to avoid contradiction for some values of the indices. One has some freedom of notation: for instance, one can eliminate case (IIa) if one includes the first three fields fields in case (IIb) and the last one in case (IIc). The relations (2.2.5) are not affected in this way.

Let us consider the set of Wick monomials $\mathcal{W}$ constructed from the free fields $A^\mu_a,\ u_a,\ \tilde{u}_a$ and Φ_a for all indices $a = 1, \dots, r$; we define the BRST operator $d_Q : \mathcal{W} \to \mathcal{W}$ as the (graded) commutator with the supercharge operator Q. Then one can prove easily that:

$$d_Q^2 = 0. \tag{2.2.8}$$

The class of observables on the factor space is defined as follows: an operator $O : \mathcal{H}^{gh,r}_{YM} \to \mathcal{H}^{gh,r}_{YM}$ induces a well defined operator $[O]$ on the factor space $\overline{Ker(Q)/Im(Q)} \simeq \mathcal{F}_m$ if and only if it verifies:

$$d_Q O|_{Ker(Q)} = 0. \tag{2.2.9}$$

Because of the relation (2.2.8) not all operators verifying the condition (2.2.9) are interesting. In fact, the operators of the type $d_Q O$ are inducing a null operator on the factor space; explicitly, we have:

$$[d_Q O] = 0. \tag{2.2.10}$$

We will construct a perturbation theory verifying Bogoliubov axioms using this set of free fields and imposing the usual axioms of causality, unitarity and relativistic invariance on the

chronological products $T(x_1,\ldots,x_n)$. Moreover, we want that the result factorizes to the physical Hilbert space in the formal adiabatic limit. This amounts to [3] - [28]:

$$d_Q T(x_1,\ldots,x_n) = i\sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T_l^\mu(x_1,\ldots,x_n) \tag{2.2.11}$$

for some auxiliary chronological products $T_l^\mu(x_1,\ldots,x_n),\quad l=1,\ldots,n$ which must be determined recurringly, together with the standard chronological products.

If one adds matter fields we proceed as before. In particular, we suppose that the BRST operator acts trivially on the matter fields. It seems that the matter field must be described by a set of Dirac fields of masses $M_A,\quad A=1,\ldots,N$ denoted by $\psi_A(x)$. These fields are characterized by the following relations [35]; here $A,B=1,\ldots,N$:

Equation of motion:

$$(i\gamma\cdot\partial + M_A)\psi_A(x) = 0. \tag{2.2.12}$$

Canonical (anti)commutation relations:

$$\{\psi_A(x),\overline{\psi_B}(y)\} = \delta_{AB} S_{M_A}(x-y) \tag{2.2.13}$$

and all other (anti)commutators are null.

By a *trivial Lagrangian* we mean a Wick expression of the type

$$L(x) = d_Q N(x) + i\frac{\partial}{\partial x^\mu} L^\mu(x) \tag{2.2.14}$$

with $L(x)$ and $L^\mu(x)$ some Wick polynomials. The first term in the previous formula gives zero by factorisation to the physical Hilbert space (according to a previous discussion) and the second one gives also zero in the adiabatic limit; this justify the elimination of such expression from the first order chronological product $T(x)$.

Let us suppose that for $|X|=1$ the expressions $T(x)$ and $T^\mu(x) = T_1^\mu(x)$ have the generic form (2.1.2):

$$T(x) = \sum c_j T_j(x) \quad T^\mu(x) = \sum c_j^\mu T_j(x) \tag{2.2.15}$$

with $c_j,\quad c_j^\mu$ some real constants. One can prove [34], [35] that the condition (2.2.11) for $n=1,2,3$ determines quite drastically the interaction Lagrangian (up to a trivial Lagrangian):

$$\begin{gathered} T(x) \equiv -f_{abc}\left[\frac{1}{2} : A_{a\mu}(x)A_{b\nu}(x)F_a^{\mu\nu}(x) :: A_a^\mu(x)u_b(x)\partial_\mu\tilde{u}_c(x) :\right], \\ +f'_{abc}\left[: \Phi_a(x)\partial_\mu\Phi_b(x)A_c^\mu(x) : -m_b : \Phi_a(x)A_{b\mu}(x)A_c^\mu(x) : -m_b : \Phi_a(x)\tilde{u}_b(x)u_c(x) :\right] \\ +f^{"}_{abc} : \Phi_a(x)\Phi_b(x)\Phi_c(x) : +j_a^\mu(x)A_{a\mu}(x) + j_a(x)\Phi_a(x) \end{gathered} \tag{2.2.16}$$

where:

$$F_a^{\mu\nu}(x) \equiv \partial^\mu A_a^\nu(x) - \partial^\nu A_a^\mu(x) \tag{2.2.17}$$

is the Yang-Mills field tensor and the so-called *currents* are:

$$j_a^\mu(x) =: \overline{\psi_A}(x)(t_a)_{AB}\gamma^\mu\psi_B(x) : + : \overline{\psi_A}(x)(t'_a)_{AB}\gamma^\mu\gamma_5\psi_B(x) : \tag{2.2.18}$$

and

$$j_a(x) =: \overline{\psi_A}(x)(s_a)_{AB}\psi_B(x) : + : \overline{\psi_A}(x)(s'_a)_{AB}\gamma_5\psi_B(x) : \tag{2.2.19}$$

where a number of restrictions must be imposed on the various constants (see [33]-[35] where the condition of gauge invariance is analysed up to order 3.)

Moreover, we can take $T^\mu(x)$ to be:

$$T^\mu(x) = f_{abc}\left[: u_a(x)A_{b\nu}(x)F_c^{\nu\mu}(x) : -\frac{1}{2} : u_a(x)u_b(x)\partial^\mu(x)\tilde{u}_c(x) :\right]$$
$$+f'_{abc}\left[m_a : A_a^\mu(x)\Phi_b(x)u_c(x) : + : \Phi_a(x)\partial^\mu\Phi_b(x)u_c(x) :\right]. + u_a(x)j_a^\mu(x). \tag{2.2.20}$$

The expressions $T(x)$ and $T^\mu(x)$ are $SL(2,\mathbb{C})$-covariant, are causally commuting and are Hermitean. Moreover we have the following ghost content:

$$gh(T(x)) = 0, \quad gh(T^\mu(x)) = 0. \tag{2.2.21}$$

Remark 2.1 *The presence of indices of type IIb and IIc is taken into account by requiring that the constants from $T(x)$ are null if one of the indices a, b, c takes such values. One can see that this does not affect the equations from the statement of the theorem.*

3 The Renormalizability of Quantum Electrodynamics

3.1 The General Setting

The case of QED is a particular case of the scheme described in the preceding Section. We have only one field of type IIb i.e. the triplet $A_\mu,\ u,\ \tilde{u}$ of null mass; they describe a system of null-mass Bosons of helicity 1 (i.e. *photons*). We also have only one Dirac field ψ describing the electron. We suppose that in the Hilbert space $\mathcal{H}^{gh}$ generated by these fields from the vacuum Ω we have a sesqui-linear form $< \cdot, \cdot >$ and we denote the conjugate of the operator O with respect to this form by $O^\dagger$. We characterize this form by requiring:

$$A_\mu(x)^\dagger = A_\mu(x), \quad u(x)^\dagger = u(x), \quad \tilde{u}(x)^\dagger = -\tilde{u}(x). \tag{3.1.1}$$

The unitary operator realizing the charge conjugation is defined by:

$$\begin{aligned} U_C A^\mu(x) U_C^{-1} = -A^\mu(x), \quad U_C u(x) U_C^{-1} = -u(x), \quad U_C \tilde{u}(x) U_C^{-1} = -\tilde{u}(x), \\ U_C \psi(x) U_C^{-1} = \gamma_0 \gamma_2 \bar{\psi}(x)^t, \quad U_C \Omega = \Omega \end{aligned} \tag{3.1.2}$$

Now, we define in $\mathcal{H}^{gh}$ the *supercharge* according to:

$$Q\Omega = 0 \tag{3.1.3}$$

and

$$\{Q, u(x)\} = 0, \quad \{Q, \tilde{u}(x)\} = -i\partial^\mu A_\mu(x), \quad [Q, A_\mu(x)] = i\partial_\mu u(x). \tag{3.1.4}$$

The expression of the BRST-operator d_Q follows as a particular case of the corresponding formulæ of the Yang-Mills case. From these properties one can derive

$$Q^2 = 0; \tag{3.1.5}$$

so we also have

$$Im(Q) \subset Ker(Q). \tag{3.1.6}$$

By definition, the interaction Lagrangian is:

$$T(x) \equiv e : \bar{\psi}(x)\gamma_\mu\psi(x) : A^\mu(x) \tag{3.1.7}$$

(here e is a real constant: the electron charge) and one can verify easily that we have the covariance properties with respect to $SL(2,\mathbb{C})$. The most important property is (2.2.11) for $n = 1$:

$$d_Q T(x) = i\frac{\partial}{\partial x^\mu} T^\mu(x) \tag{3.1.8}$$

with:

$$T^\mu(x) \equiv e : \bar{\psi}(x)\gamma^\mu\psi(x) : u(x). \tag{3.1.9}$$

One can easily check that we have charge-conjugation invariance in the sense:

$$U_C T(x) U_C^{-1} = T(x), \quad U_C T^\mu(x) U_C^{-1} = T^\mu(x). \tag{3.1.10}$$

We note that we also have:

$$d_Q T^\mu(x) = 0. \tag{3.1.11}$$

3.2 The Main Result

It is convenient to write the formulæ (3.1.8) and (3.1.11) in a compact way as follows. One denotes by $A^k(x),\ k = 1,\dots,5$ the expressions $T(x),\ T^\mu(x)$; that is, the index i can take the values $L,\quad \mu$ according to the identification: $A^L(x) \equiv T(x),\quad A^\mu(x) \equiv T^\mu(x)$. Then we can write the preceding gauge invariance conditions in the form:

$$d_Q A^k(x) = i \sum_{m=1}^{5} c_m^{k;\mu} \frac{\partial}{\partial x^\mu} A^m(x), \quad k = 1,\dots,5 \tag{3.2.1}$$

for some constants $c_m^{k;\mu}$; the explicit expressions can be obtained from the corresponding gauge conditions. Only the expression

$$c_\nu^{L;\mu} \equiv \delta_\nu^\mu \tag{3.2.2}$$

are non-zero. Then we can prove the following result:

Theorem 3.1 *One can chose the chronological products such that, beside the fulfilment of the Bogoliubov axioms, the following identities are verified:*

$$d_Q T(A^{k_1}(x_1),\dots,A^{k_p}(x_p)) = i \sum_{l=1}^{p} (-1)^{s_l} \sum_m c_m^{k_l;\mu} \frac{\partial}{\partial x_l^\mu} T(A^{k_1}(x_1),\dots,A^m(x_l),\dots,A^{k_p}(x_p)) \tag{3.2.3}$$

for all $p \in \mathbb{N}$ and all $k_1,\dots,k_p = 1,\dots,5$. Here we have denoted

$$s_0 \equiv 0, \quad s_l \equiv \sum_{j=1}^{l-1} gh(A_j), \quad \forall l = 1,\dots,p. \tag{3.2.4}$$

Proof: (i) We use induction. Suppose we have constructed the chronological products such that that all conditions are satisfied up to order $p = n-1$. One can construct the chronological products in order n such that all Bogoliubov axioms are satisfied, except the condition of gauge invariance. This can be done directly from the Epstein-Glaser methods [36] or using the extension method [21]. One can choose the chronological products to depend only on the fields

$$A_\mu,\ u,\ \psi,\ \bar{\psi} \tag{3.2.5}$$

and such that

$$gh(T(A^{k_1}(x_1),\dots,A^{k_p}(x_p))) = \sum_{l=1}^{n} gh(A^{k_l}). \tag{3.2.6}$$

Moreover, the symmetry axiom implies relations of the type:

$$T(A_1(x_1), A_2(x_2),\dots) = (-1)^{gh(A_1)gh(A_2)} T(A_2(x_2), A_1(x_1),\dots). \tag{3.2.7}$$

From the induction procedure (or the extension method) one can easily prove the the possible obstructions to the gauge invariance condition (3.2.3) in order n have a particular structure.

We have:

$$
\begin{aligned}
& d_Q T(A^{k_1}(x_1), \ldots, A^{k_n}(x_n)) \\
= i \sum_{l=1}^{n} (-1)^{s_l} \sum_m c_m^{k_l;\mu} \frac{\partial}{\partial x_l^\mu} & T(A^{k_1}(x_1), \ldots, A^m(x_l), \ldots, A^{k_n}(x_n)) \\
& + P^{k_1,\ldots,k_n}(x_1, \ldots, x_n)
\end{aligned}
\tag{3.2.8}
$$

where $P^{\cdots}(X) \equiv P^{\cdots}(x_1, \ldots, x_n)$ are quasi-local operators called *anomalies*. They have the following structure:

$$
P(X) = \sum_L \left[p_L(\partial)\delta(X)\right] W_L(X) \tag{3.2.9}
$$

where W_L are Wick monomials and p_L are polynomials in the derivatives of the type

$$
p_L(X) = \sum_{|\alpha| \leq deg(p_L)} c_{L,\alpha} \partial^\alpha \tag{3.2.10}
$$

with the maximal degree restricted by

$$
deg(p_L) + \omega_L \leq 5. \tag{3.2.11}
$$

Moreover, we can easily obtain:

$$
gh(P^{k_1,\ldots,k_n}(X)) = \sum_{l=1}^{n} gh(A^{k_l}) + 1. \tag{3.2.12}
$$

Finally, the anomalies can be chosen $SL(2, \mathbb{C})$-covariant and charge conjugation invariant:

$$
U_C P^{k_1,\ldots,k_n}(X) U_C^{-1} = P^{k_1,\ldots,k_n}(X). \tag{3.2.13}
$$

(ii) We have a lot of restrictions on the anomalies. The most sever one comes from from (3.2.11) and (3.2.12): we obtain that for

$$
\sum_{l=1}^{n} gh(A^{k_l}) \geq 5 \tag{3.2.14}
$$

there are no anomalies. From this restriction it follows that we have the following set of relations with possible anomalies:

$$
\begin{aligned}
& d_Q T(T(x_1), \ldots, T(x_n)) \\
= i \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} & T(T(x_1), \ldots, T^\mu(x_l), \ldots, T(x_n)) + P_1(x_1, \ldots, x_n)
\end{aligned}
\tag{3.2.15}
$$

$$
\begin{aligned}
& d_Q T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) \\
= -i \sum_{l=2}^{n} \frac{\partial}{\partial x_l^\nu} & T(T^\mu(x_1), T(x_2), \ldots, T^\nu(x_l), \ldots, T(x_n)) \\
& + P_2^\mu(x_1, \ldots, x_n)
\end{aligned}
\tag{3.2.16}
$$

$$
\begin{aligned}
& d_Q T(T^\mu(x_1), T^\nu(x_2), T(x_3), \dots, T(x_n)) \\
= i \sum_{l=3}^{n} \frac{\partial}{\partial x_l^\rho} & T(T^\mu(x_1), T^\nu(x_2), T(x_3), \dots, T^\rho(x_l), \dots, T(x_n)) \\
& + P_3^{\mu\nu}(x_1, \dots, x_n)
\end{aligned}
\tag{3.2.17}
$$

$$
\begin{aligned}
& d_Q T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \dots, T(x_n)) \\
= -i \sum_{l=4}^{n} \frac{\partial}{\partial x_l^\sigma} & T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \dots, T^\sigma(x_l), \dots, T(x_n)) \\
& + P_4^{\mu\nu\rho}(x_1, \dots, x_n)
\end{aligned}
\tag{3.2.18}
$$

$$
\begin{aligned}
& d_Q T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), \dots, T(x_n)) \\
= i \sum_{l=5}^{n} \frac{\partial}{\partial x_l^\lambda} & T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), \dots, T^\lambda(x_l), \dots, T(x_n)) \\
& + P_5^{\mu\nu\rho\lambda}(x_1, \dots, x_n)
\end{aligned}
\tag{3.2.19}
$$

where we use, as before, the convention $\sum_\emptyset \equiv 0$. We can assume that:

$$
P_3^{\mu\nu}(X) = 0, \quad |X| = 1, \quad P_4^{\mu\nu\rho}(X) = 0, \quad |X| = 2, \quad P_5^{\mu\nu\rho}(X) = 0, \quad |X| = 3 \tag{3.2.20}
$$

without losing generality. The anomalies verify the restrictions (3.2.12) and (3.2.11) and they depend only on the fields

$$
A_\mu,\ \partial_\nu A_\mu,\ u,\ \partial_\nu u,\ \psi,\ \partial_\nu \psi, \bar{\psi},\ \partial_\nu \bar{\psi} \tag{3.2.21}
$$

In fact, we can refine the induction hypothesis: we can assume that $T(T(x_1), \dots, T(x_n))$ does not depend on u, $T(T^\mu(x_1), T(x_2), \dots, T(x_n))$ depends linearly on $u(x_1)$ and does not depend on $A_\mu(x_1)$, $T(T^\mu(x_1), T^\nu(x_2), T(x_3), \dots, T(x_n))$ depends linearly on $: u(x_1) u(x_2) :$ and does not depend on $A_\mu(x_i), \quad i = 1, 2$, etc.

Then it follows that the anomalies depend on the fields

$$
A_\mu,\ u,\ \partial_\nu u,\ \psi,\ \partial_\nu \psi, \bar{\psi},\ \partial_\nu \bar{\psi}. \tag{3.2.22}
$$

From (3.2.7), we get the following symmetry properties:

$$
P_1(x_1, \dots, x_n) \quad \text{is symmetric in} \quad x_1, \dots, x_n; \tag{3.2.23}
$$

$$
P_2^\mu(x_1, \dots, x_n) \quad \text{is symmetric in} \quad x_2, \dots, x_n; \tag{3.2.24}
$$

$$
P_3^{\mu\nu}(x_1, \dots, x_n) \quad \text{is symmetric in} \quad x_3, \dots, x_n; \tag{3.2.25}
$$

$$
P_4^{\mu\nu\rho}(x_1, \dots, x_n) \quad \text{is symmetric in} \quad x_4, \dots, x_n; \tag{3.2.26}
$$

$$
P_5^{\mu\nu\rho\sigma}(x_1, \dots, x_n) \quad \text{is symmetric in} \quad x_5, \dots, x_n; \tag{3.2.27}
$$

$$P_3^{\mu\nu}(x_1,\ldots,x_n) \quad \text{is antisymmetric in} \quad (x_1,\mu),(x_2,\nu); \tag{3.2.28}$$

$$P_4^{\mu\nu\rho}(x_1,\ldots,x_n) \quad \text{is antisymmetric in} \quad (x_1,\mu),(x_2,\nu),(x_3,\rho); \tag{3.2.29}$$

$$P_5^{\mu\nu\rho\sigma}(x_1,\ldots,x_n) \quad \text{is antisymmetric in} \quad (x_1,\mu),(x_2,\nu),(x_3,\rho),(x_4,\sigma). \tag{3.2.30}$$

(iii) If we apply the operator d_Q to the anomalous relations (3.2.15)-(3.2.19) we easily obtain some consistency relations quite analogous to the well-known Wess-Zumino consistency relations:

$$d_Q P_1(x_1,\ldots,(x_n) = i\sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} P_2^\mu(x_l,x_1,\ldots,\hat{x}_l,\ldots,x_n) \tag{3.2.31}$$

$$d_Q P_2^\mu(x_1),\ldots,(x_n) = -i\sum_{l=2}^{n} \frac{\partial}{\partial x_l^\nu} P_3^{\mu\nu}(x_1,x_l,x_2,\ldots,\hat{x}_l,\ldots,x_n) \tag{3.2.32}$$

$$d_Q P_3^{\mu\nu}(x_1,\ldots,x_n) = i\sum_{l=3}^{n} \frac{\partial}{\partial x_l^\rho} P_4^{\mu\nu\rho}(x_1,x_2,x_l,x_3,\ldots,\hat{x}_l,\ldots,x_n) \tag{3.2.33}$$

$$d_Q P_4^{\mu\nu\rho}(x_1,x_n) = -i\sum_{l=4}^{n} \frac{\partial}{\partial x_l^\sigma} P_5^{\mu\nu\rho\sigma}(x_1,x_2,x_3,x_l,x_4,\ldots,\hat{x}_l,\ldots,x_n) \tag{3.2.34}$$

$$d_Q P_5^{\mu\nu\rho\sigma}(x_1,\ldots,x_n) = 0. \tag{3.2.35}$$

We will use repeatedly the identity

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_l^\rho} \delta(X) = 0 \tag{3.2.36}$$

where

$$\delta(X) \equiv \delta(x_1 - x_n)\cdots\delta(x_{n-1} - x_n). \tag{3.2.37}$$

If we take into account (3.2.11) and (3.2.12), the generic form of the anomalies is:

$$\begin{aligned} P_l^{\cdots}(X) = {} & \delta(X)\widetilde{W}_l^{\cdots}(x_1) + \sum_{p=1}^{n} \left[\frac{\partial}{\partial x_p^\mu}\delta(X)\right] \widetilde{W}_{l;p}^{\ldots;\mu}(X) + \sum_{p,q=1}^{n} \left[\frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu}\delta(X)\right] \widetilde{W}_{l;pq}^{\ldots;\mu\nu}(X) \\ & + \sum_{p,q,r=1}^{n} \left[\frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho}\delta(X)\right] \widetilde{W}_{l;pqr}^{\ldots;\mu\nu\rho}(X) + \sum_{p,q,r,s=1}^{n} \left[\frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho \partial x_s^\sigma}\delta(X)\right] \widetilde{W}_{l;pqrs}^{\ldots;\mu\nu\rho\sigma}(X) \end{aligned} \tag{3.2.38}$$

where $\widetilde{W}_{\ldots}^{\cdots}$ are some Wick polynomials with convenient symmetry properties. If we use (3.2.36) we can eliminate all derivatives with respect to one variable, say x_1 if we redefine conveniently the expressions $\widetilde{W}_{\ldots}^{\cdots}$:

$$\begin{aligned} P_l^{\cdots}(X) = {} & \delta(X)W_l^{\cdots}(x_1) + \sum_{p=2}^{n} \frac{\partial}{\partial x_p^\mu}\delta(X) W_{l;p}^{\ldots;\mu}(x_1) + \sum_{p,q=2}^{n} \frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu}\delta(X) W_{l;pq}^{\ldots;\mu\nu}(x_1) \\ & + \sum_{p,q,r=2}^{n} \frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho}\delta(X) W_{l;pqr}^{\ldots;\mu\nu\rho}(x_1) + \sum_{p,q,r,s=2}^{n} \frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho \partial x_s^\sigma}\delta(X) W_{l;pqrs}^{\ldots;\mu\nu\rho\sigma}(x_1) \end{aligned} \tag{3.2.39}$$

If $f \in \mathcal{S}(\mathbb{R}^{4n})$ is arbitrary we have:

$$< P_l^{\cdots}, f(X) >= \int dx f(x,\ldots,x) W_l^{\cdots}(x) - \sum_{p=2}^n \int dx (\partial_\mu^p f)(x,\ldots,x) W_{l;p}^{\cdots;\mu}(x)$$
$$+ \sum_{p,q=2}^n \int dx (\partial_\mu^p \partial_\nu^q f)(x,\ldots,x) W_{l;pq}^{\cdots;\mu\nu}(x) + \cdots \tag{3.2.40}$$

But the expressions $f(x,\ldots,x)$, $(\partial_\mu^p f)(x,\ldots,x)$, $(\partial_\mu^p \partial_\nu^q f)(x,\ldots,x), \ldots \quad p,q,\ldots \geq 2$ can be chosen arbitrary, so we have:

$$P_l^{\cdots}(X) \quad \Longleftrightarrow \quad W_l^{\cdots} = 0, \quad W_{l;p}^{\cdots;\mu}(X) = 0, \quad W_{l;pq}^{\cdots;\mu\nu}(X) = 0, \quad ,\ldots \tag{3.2.41}$$

As a consequence, every symmetry property

$$< P_l^{\cdots}(X), f^g(X) >=< P_l^{\cdots}(X), f(X) > \tag{3.2.42}$$

for g an arbitrary symmetry, will be equivalent to corresponding symmetry properties for the Wick polynomials:

$$g \cdot W = W. \tag{3.2.43}$$

(iv) Let us consider $l = 3,4,5$; because in this case $gh(P_l^{\cdots}) \geq 3$ in every Wick polynomial $W_{\cdots}^{\cdots}$ from (3.2.39) we have at least two factors u (the third can be ∂u) so we get:

$$P_l^{\cdots}(X) = 0, \quad l = 3,4,5. \tag{3.2.44}$$

The generic expression of P_2 is:

$$P_2^\mu(X) = \delta(X) W_2^\mu(x_1) + \sum_{p=2}^n \frac{\partial}{\partial x_p^\nu} \delta(X) W_{2;p}^{\mu;\nu}(x_1) + \sum_{p,q=2}^n \frac{\partial^2}{\partial x_p^\nu \partial x_q^\rho} \delta(X) W_{2;pq}^{\mu;\nu\rho}(x_1)$$
$$+ \sum_{p,q,r=2}^n \frac{\partial^3}{\partial x_p^\nu \partial x_q^\rho \partial x_r^\sigma} \delta(X) W_{2;pqr}^{\mu;\nu\rho\sigma}(x_1) \tag{3.2.45}$$

Because $gh(P_2) = 2$ we have $W_{l;pqr}^{\mu;\nu\rho\sigma} \sim: uu := 0$ so the last term disappears. If we use the symmetry property (3.2.24) we get:

$$W_{2;p}^{\mu;\nu} = W_{2;2}^{\mu;\nu} \equiv W_2^{\mu;\nu}, \quad \forall p = 2,\ldots,n,$$
$$W_{2;pq}^{\mu;\nu\rho} = W_{2;22}^{\mu;\nu\rho} \equiv W_2^{\mu;\nu\rho}, \quad \forall p,q = 2,\ldots,n \tag{3.2.46}$$

so we can write the preceding expression more simply:

$$P_2^\mu(X) = \delta(X) W_2^\mu(x_1) + \sum_{p=2}^n \frac{\partial}{\partial x_p^\nu} \delta(X) W_2^{\mu;\nu}(x_1) + \sum_{p,q=2}^n \frac{\partial^2}{\partial x_p^\nu \partial x_q^\rho} \delta(X) W_2^{\mu;\nu\rho}(x_1). \tag{3.2.47}$$

If we use (3.2.36) we obtain after some relabelling:

$$P_2^\mu(X) = \delta(X)W_2^\mu(x_1) + \frac{\partial}{\partial x_1^\nu}\left[\delta(X)W_2^{\mu;\nu}(x_1)\right] + \frac{\partial^2}{\partial x_1^\nu \partial x_1^\rho}\left[\delta(X)W_2^{\mu;\nu\rho}(x_1)\right] \tag{3.2.48}$$

and we can assume that

$$W_2^{\mu;\nu\rho} = (\nu \leftrightarrow \rho). \tag{3.2.49}$$

Because $gh(P_1) = 1$ we have the generic expression:

$$P_1(X) = \delta(X)W_1(x_1) + \sum_{p=2}^{n} \frac{\partial}{\partial x_p^\mu}\delta(X)W_{1;p}^\mu(x_1) + \sum_{p,q=2}^{n} \frac{\partial^2}{\partial x_p^\mu \partial x_q^\nu}\delta(X)W_{1;pq}^{\mu\nu}(x_1)$$
$$+ \sum_{p,q,r=2}^{n} \frac{\partial^3}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho}\delta(X)W_{1;pqr}^{\mu\nu\rho}(x_1) + \sum_{p,q,r,s=2}^{n} \frac{\partial^4}{\partial x_p^\mu \partial x_q^\nu \partial x_r^\rho \partial x_s^\sigma}\delta(X)W_{1;pqrs}^{\mu\nu\rho\sigma}(x_1) \tag{3.2.50}$$

The symmetry requirement (3.2.23) in $x_2, \ldots, x_n$ leads as above at a simpler form:

$$P_1(X) = \delta(X)W_1(x_1) + \frac{\partial}{\partial x_1^\mu}\left[\delta(X)W_1^\mu(x_1)\right] + \frac{\partial^2}{\partial x_1^\mu \partial x_1^\nu}\left[\delta(X)W_1^{\mu\nu}(x_1)\right]$$
$$+ \frac{\partial^3}{\partial x_1^\mu \partial x_1^\nu \partial x_1^\rho}\left[\delta(X)W_1^{\mu\nu\rho}(x_1)\right] + \frac{\partial^4}{\partial x_1^\mu \partial x_1^\nu \partial x_1^\rho \partial x_1^\sigma}\left[\delta(X)W_1^{\mu\nu\rho\sigma}(x_1)\right] \tag{3.2.51}$$

and the Wick polynomials have convenient symmetry properties. We have the generic form

$$W_1^{\mu\nu\rho\sigma} = c_1^{\mu\nu\rho\sigma}u \tag{3.2.52}$$

with $c_1^{\mu\nu\rho\sigma}$ a Lorentz covariant tensor. If we perform the finite renormalization:

$$T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) \rightarrow T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) + i\frac{\partial^3}{\partial x_1^\nu \partial x_1^\rho \partial x_1^\sigma}\left[\delta(X)W_1^{\mu\nu\rho\sigma}(x_1)\right] \tag{3.2.53}$$

we do not affect the symmetry properties and the field dependence (3.2.22) but as a result we eliminate the last term in the expression of P_1. We impose now the symmetry property (3.2.23) in x_1, x_2 and obtain that in fact:

$$P_1(X) = \delta(X)W_1(x_1). \tag{3.2.54}$$

(v) Next, we use the consistency conditions (3.2.31)-(3.2.35). Only the first two one are non-trivial. We get the following conditions:

$$d_Q W_1 = i\partial_\mu W_2^\mu, \quad W_2^{\mu;\nu} = -(\mu \leftrightarrow \nu), \quad W_2^{\mu;\nu\rho} + W_2^{\nu;\mu\rho} + W_2^{\rho;\mu\nu} = 0 \tag{3.2.55}$$

and respectively:

$$d_Q W_2^\mu = 0, \quad d_Q W_2^{\mu;\nu} = 0, \quad d_Q W_2^{\mu;\nu\rho} = 0. \tag{3.2.56}$$

We can still simplify the expressions of the anomalies by finite renormalizations. We present briefly the details. The generic form of $W_2^{\mu;\nu\rho}$ is

$$W_2^{\mu;\nu\rho} = c_2^{\mu\nu\rho\sigma} : u\partial_\sigma u : \tag{3.2.57}$$

with $c_2^{\mu\nu\rho\sigma}$ a Lorentz invariant tensor. If we define

$$U_2^{\mu;\nu\rho} = c_2^{\mu\nu\rho\sigma} : uA_\sigma : \tag{3.2.58}$$

then we have:

$$d_Q U_2^{\mu;\nu\rho} = -iW_2^{\mu;\nu\rho}. \tag{3.2.59}$$

It follows that if we perform the finite renormalization:

$$T(T^\mu(x_1), T(x_2), \dots, T(x_n)) \to T(T^\mu(x_1), T(x_2), \dots, T(x_n)) + i\frac{\partial^2}{\partial x_1^\nu \partial x_1^\rho}\left[\delta(X)U_2^{\mu\nu\rho}(x_1)\right] \tag{3.2.60}$$

we do not change the symmetry properties and the field structure; moreover we do not enter in conflict with (3.2.53). As a result we make

$$W_2^{\mu;\nu\rho} = 0. \tag{3.2.61}$$

In the same way, we have the generic expression:

$$W_2^{\mu;\nu} = \tilde{c}_2^{\mu\nu\rho\sigma} : u\partial_\rho u A_\sigma : \tag{3.2.62}$$

with $\tilde{c}_2^{\mu\nu\rho\sigma}$ a Lorentz invariant tensor. From the second equation (3.2.55) we obtain antisymmetry in the first two indices and from the second equation (3.2.56) we get symmetry in the last two indices. All these restrictions lead to

$$\tilde{c}_2^{\mu\nu\rho\sigma} = 0. \tag{3.2.63}$$

So, in the end we have:

$$P_2^\mu = \delta(X)W_2^\mu(x_1). \tag{3.2.64}$$

From the equations (3.2.55) and (3.2.56) we are left with:

$$d_Q W_1 = i\partial_\mu W_2^\mu, \quad d_Q W_2^\mu = 0. \tag{3.2.65}$$

(vi) We have now the generic form:

$$W_1^\mu = d_1 : u\partial^\mu u : + d_2 : u\partial^\mu u A_\rho A^\rho : + d_3 : u\partial_\rho u A_\rho A^\mu : \tag{3.2.66}$$

for some constants d_i. The second equation (3.2.65) gives $d_3 = 2d_1$. If we define:

$$U_2^\mu = d_1 : uA^\mu : + d_2 : uA^\mu u A_\rho A^\rho : \tag{3.2.67}$$

we get

$$d_Q U_2^\mu = iW_2^\mu. \tag{3.2.68}$$

Now we perform the finite renormalization

$$T(T^{\mu}(x_1), T(x_2), \ldots, T(x_n)) \rightarrow T(T^{\mu}(x_1), T(x_2), \ldots, T(x_n)) + i\delta(X) U_2^{\mu}(x_1) \tag{3.2.69}$$

we do not affect the properties of the chronological products, we do not spoil the previous two renormalizations and we make:

$$P_2^{\mu} = 0. \tag{3.2.70}$$

It follows that we still have to impose:

$$d_Q W_1 = 0. \tag{3.2.71}$$

The generic form of W_1 is:

$$\begin{aligned} W_1 = c_1 u + c_2 :u A_{\mu} A^{\mu}: + c_3 :\partial_{\mu} u A^{\mu}: + c_4 :u \bar{\psi} \psi: + c_5 :u \bar{\psi} \gamma_5 \psi: \\ + c_6 :u A_{\mu} \bar{\psi} \gamma^{\mu} \psi: + c_7 :u A_{\mu} \bar{\psi} \gamma^{\mu} \gamma_5 \psi: + c_8 :\partial_{\mu} u A^{\mu} A_{\rho} A^{\rho}: + c_9 :u A^{\mu} A^{\mu} A_{\rho} A^{\rho}: \end{aligned} \tag{3.2.72}$$

Now it is time to use charge conjugation invariance of the anomalies (3.2.13) for P_1; we get easily: $c_i = 0, \quad i = 1, 2, 4, 5, 7, 9$. If we impose the condition (3.2.71) we get $c_6 = 0$. It follows that we are left with:

$$W_1 = c_3 :\partial_{\mu} u A^{\mu}: + c_9 :\partial_{\mu} u A^{\mu} A_{\rho} A^{\rho}: \tag{3.2.73}$$

If we define:

$$U_1 \equiv \frac{1}{2} c_3 :A_{\mu} A^{\mu}: + \frac{1}{4} c_9 :A_{\mu} A^{\mu} A_{\rho} A^{\rho}: \tag{3.2.74}$$

we have:

$$d_Q U_1 = i W_1 \tag{3.2.75}$$

Finally, we preform the finite renormalization:

$$T(T(x_1), \ldots, T(x_n)) \rightarrow T(T(x_1), \ldots, T(x_n)) + i\delta(X) U_1(x_1) \tag{3.2.76}$$

we do not affect the symmetry properties and the field structure (3.2.22). As a result we get:

$$P_1(X) = 0 \tag{3.2.77}$$

and the proof is finished. ■

Remark 3.2 *It is easy to see that the same pattern works for scalar electrodynamics also. A minor modification appears for the expression of W_1: the terms $c_4 - c_7$ must be replaced by:*

$$W_1 = c_4 :u \bar{\phi} \phi: + c_5 :u A_{\mu} \bar{\phi} \partial^{\mu} \phi: + c_6 :u A_{\mu} \partial^{\mu} \bar{\phi} \phi: \tag{3.2.78}$$

The first contribution is cancelled by charge conjugation invariance and the last two by the condition (3.2.71).

4 The Structure of the Anomalies in Higher Orders

4.1 The Anomalous Gauge Equations

We give now the results for the Yang-Mills model as presented in Subsection 2.2. By comparison to the case of QED, two important modification appear. The first one is the relation (3.1.11) which is replaced by:

$$d_Q T^\mu(x) = i \frac{\partial}{\partial x^\nu} T^{\mu\nu}(x) \tag{4.1.1}$$

where:

$$T^{\mu\nu}(x) \equiv \frac{1}{2} f_{abc} : u_a u_b F_c^{\mu\nu} : \tag{4.1.2}$$

Let us note the antisymmetry property:

$$T^{\mu\nu}(x) = -T^{\nu\mu}(x) \tag{4.1.3}$$

and the analogue of (3.1.11):

$$d_Q T^{\mu\nu}(x) = 0. \tag{4.1.4}$$

We also have:

$$gh(T^{\mu\nu}(x)) = 2. \tag{4.1.5}$$

The second change is the disappearance of charge conjugation invariance. Because of these changes we will not be able to prove the disappearance of the anomalies in higher orders of perturbation theory. Instead, we will be able to give the generic structure of these anomalies. The computations are similar to those from the preceding Section but are more complicated from the combinatorial point of view. Because there are no essential new subtleties we will give only the results.

Like in the case of QED we write the formulæ (3.1.8), (4.1.1) and (4.1.4) in a compact way as follows. One denotes by $A^k(x),\ k = 1,\ldots,11$ the expressions $T(x),\ T^\mu(x),\ T^{\mu\nu}$; the index i can take the values $L, \quad \mu, \quad \mu\nu$ according to the identifications $A^L(x) \equiv T(x), \quad A^\mu(x) \equiv T^\mu(x), \quad A^{\mu\nu}(x) \equiv T^{\mu\nu}(x)$. Then we can write the gauge invariance conditions in the form (3.2.1):

$$d_Q A^k(x) = i \sum_m c_m^{k;\mu} \frac{\partial}{\partial x^\mu} A^m(x), \quad k = 1,\ldots,11 \tag{4.1.6}$$

for some constants $c_m^{k;\mu}$; the explicit expressions are:

$$c_\nu^{L;\mu} \equiv \delta_\nu^\mu, \quad c_{\rho\sigma}^{\nu;\mu} \equiv \frac{1}{2}\left(\delta_\rho^\nu \delta_\sigma^\mu - \delta_\rho^\mu \delta_\sigma^\nu\right) \tag{4.1.7}$$

and the others are zero. Then we conjecture the following result: one can chose the chronological products such that, beside the fulfilment of the Bogoliubov axioms, the following identities are verified:

$$d_Q T(A^{k_1}(x_1),\ldots,A^{k_p}(x_p)) = i \sum_{l=1}^p (-1)^{s_l} \sum_m c_m^{k_l;\mu} \frac{\partial}{\partial x_l^\mu} T(A^{k_1}(x_1),\ldots,A^m(x_l),\ldots,A^{k_p}(x_p)) \tag{4.1.8}$$

for all $p \in \mathbb{N}$ and all $k_1, \dots, k_p = 1, \dots, 11$. Here the expression s_l has the same significance as in the case of QED.

There are a number of facts which can be proved identically. First one can prove by induction that one can choose the chronological products such that one has (3.2.6), the symmetry property (3.2.7) and

$$T(T^{\mu\nu}(x_1), A_2(x_2), \dots, A_n(x_n)) = -T(T^{\nu\mu}(x_1), A_2(x_2), \dots, A_n(x_n)). \tag{4.1.9}$$

Next, we can prove that the chronological product can be chosen to depend on the following fields: are build only from the fields:

$$A^\mu_a, F^{\mu\nu}_a, u_a, \tilde{u}_a, \partial_\mu \tilde{u}_a, \Phi_a, \partial_\mu \Phi_a, \psi_A, \overline{\psi}_A. \tag{4.1.10}$$

Suppose that we have proved the identity (3.2.3) up to the order $n-1$; then in order in order n we must have a relation of the type (3.2.8) where $P_{\dots}(X) \equiv P_{\dots}(x_1, \dots, x_n)$ are the anomalies having the structure (3.2.9). The maximal degree of the anomaly is also restricted by (3.2.11) and we still have the constraint (3.2.12) coming from the ghost number counting. The anomalies will depend on the following set of fields:

$$A^\mu_a, \partial_\mu A^\nu_a, \partial_\rho F^{\mu\nu}_a, u_a, \partial_\mu u_a, \tilde{u}_a, \partial_\mu \tilde{u}_a, \partial_\mu \partial_\nu \tilde{u}_a, \Phi_a, \partial_\mu \Phi_a, \partial_\mu \partial_\nu \Phi_a, \psi_A, \partial_\mu \psi_A, \overline{\psi}_A, \partial_\mu \overline{\psi}_a \tag{4.1.11}$$

and the factor $\partial_\mu u_a$ can appear only once in any Wick term of the anomaly. Finally, the anomalies can be chosen $SL(2,\mathbb{C})$-covariant.

From the restrictions (3.2.11) and (3.2.12) we obtain that the possible anomalies can appear in the following relations:

$$d_Q T(T(x_1), \dots, T(x_n)) =$$
$$i \sum_{l=1}^n \frac{\partial}{\partial x^\mu_l} T(T(x_1), \dots, T^\mu(x_l), \dots, T(x_n)) + P_1(x_1, \dots, x_n) \tag{4.1.12}$$

$$d_Q T(T^\mu(x_1), T(x_2), \dots, T(x_n)) = i \frac{\partial}{\partial x^\mu_1} T(T^{\mu\nu}(x_1), T(x_2), \dots, T(x_n))$$
$$-i \sum_{l=2}^n \frac{\partial}{\partial x^\nu_l} T(T^\mu(x_1), T(x_2), \dots, T^\nu(x_l), \dots, T(x_n)) + P^\mu_2(x_1, \dots, x_n) \tag{4.1.13}$$

$$d_Q T(T^\mu(x_1), T^\nu(x_2), T(x_3), \dots, T(x_n)) =$$
$$i \frac{\partial}{\partial x^\rho_1} T(T^{\mu\rho}(x_1), T^\nu(x_2), T(x_3), \dots, T(x_n)) - i \frac{\partial}{\partial x^\rho_2} T(T^\mu(x_1), T^{\nu\rho}(x_2), T(x_3), \dots, T(x_n))$$
$$+i \sum_{l=3}^n \frac{\partial}{\partial x^\rho_l} T(T^\mu(x_1), T^\nu(x_2), T(x_3), \dots, T^\rho(x_l), \dots, T(x_n)) + P^{\mu\nu}_3(x_1, \dots, x_n) \tag{4.1.14}$$

$$
\begin{aligned}
&d_Q T(T^{\mu\nu}(x_1), T(x_2), \ldots, T(x_n)) = \\
&i \sum_{l=2}^{n} \frac{\partial}{\partial x_l^\rho} T(T^{\mu\nu}(x_1), T(x_2), \ldots, T^\rho(x_l), \ldots, T(x_n)) + P_4^{\mu\nu}(x_1, \ldots, x_n)
\end{aligned} \tag{4.1.15}
$$

$$
\begin{aligned}
&d_Q T(T^{\mu\nu}(x_1), T^\rho(x_2), T(x_3), \ldots, T(x_n)) = \\
&i \frac{\partial}{\partial x_2^\sigma} T(T^{\mu\nu}(x_1), T^{\rho\sigma}(x_2), T(x_3), \ldots, T(x_n)) \\
&-i \sum_{l=3}^{n} \frac{\partial}{\partial x_l^\sigma} T(T^{\mu\nu}(x_1), T^\rho(x_2), \ldots, T^\sigma(x_l), \ldots, T(x_n)) + P_5^{\mu\nu\rho}(x_1, \ldots, x_n)
\end{aligned} \tag{4.1.16}
$$

$$
\begin{aligned}
&d_Q T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \ldots, T(x_n)) = \\
&i \frac{\partial}{\partial x_1^\sigma} T(T^{\mu\sigma}(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \ldots, T(x_n)) \\
&-i \frac{\partial}{\partial x_2^\sigma} T(T^\mu(x_1), T^{\nu\sigma}(x_2), T^\rho(x_3), T(x_4), \ldots, T(x_n)) \\
&+i \frac{\partial}{\partial x_3^\sigma} T(T^\mu(x_1), T^\nu(x_2), T^{\rho\sigma}(x_3), T(x_4), \ldots, T(x_n)) \\
&-i \sum_{l=4}^{n} \frac{\partial}{\partial x_l^\sigma} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \ldots, T^\sigma(x_l), \ldots, T(x_n)) \\
&+P_6^{\mu\nu\rho}(x_1, \ldots, x_n)
\end{aligned} \tag{4.1.17}
$$

$$
\begin{aligned}
&d_Q T(T^{\mu\nu}(x_1), T^{\rho\sigma}(x_2), T(x_3), \ldots, T(x_n)) = \\
&i \sum_{l=3}^{n} \frac{\partial}{\partial x_l^\lambda} T(T^{\mu\nu}(x_1), T^{\rho\sigma}(x_2), T(x_3), \ldots, T^\lambda(x_l), \ldots, T(x_n)) \\
&+P_7^{\mu\nu\rho\sigma}(x_1, \ldots, x_n)
\end{aligned} \tag{4.1.18}
$$

$$
\begin{aligned}
&d_Q T(T^{\mu\nu}(x_1), T^\rho(x_2), T^\sigma(x_3), T(x_4), \ldots, T(x_n)) = \\
&i \frac{\partial}{\partial x_2^\lambda} T(T^{\mu\nu}(x_1), T^{\rho\lambda}(x_2), T^\sigma(x_3), T(x_4), \ldots, T(x_n)) \\
&-i \frac{\partial}{\partial x_3^\lambda} T(T^{\mu\nu}(x_1), T^\rho(x_2), T^{\sigma\lambda}(x_3), T(x_4), \ldots, T(x_n)) \\
&+i \sum_{l=4}^{n} \frac{\partial}{\partial x_l^\lambda} T(T^{\mu\nu}(x_1), T^\rho(x_2), T^\sigma(x_3), T(x_4), \ldots, T^\lambda(x_l), \ldots, T(x_n)) \\
&+P_8^{\mu\nu\rho\sigma}(x_1, \ldots, x_n)
\end{aligned} \tag{4.1.19}
$$

$$
d_Q T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), \ldots, T(x_n)) =
$$

$$
\begin{aligned}
& i\frac{\partial}{\partial x_1^\lambda}T(T^{\mu\lambda}(x_1),T^\nu(x_2),T^\rho(x_3),T^\sigma(x_4),T(x_5),\ldots,T(x_n)) \\
& -i\frac{\partial}{\partial x_2^\lambda}T(T^{\mu}(x_1),T^{\nu\lambda}(x_2),T^\rho(x_3),T^\sigma(x_4),T(x_5),\ldots,T(x_n)) \\
& +i\frac{\partial}{\partial x_3^\lambda}T(T^{\mu}(x_1),T^{\nu}(x_2),T^{\rho\lambda}(x_3),T^\sigma(x_4),T(x_5),\ldots,T(x_n)) \\
& -i\frac{\partial}{\partial x_4^\lambda}T(T^{\mu}(x_1),T^{\nu}(x_2),T^{\rho}(x_3),T^{\sigma\lambda}(x_4),T(x_5),\ldots,T(x_n)) \\
& +i\sum_{l=5}^n\frac{\partial}{\partial x_l^\lambda}T(T^{\mu}(x_1),T^{\nu}(x_2),T^{\rho}(x_3),T^{\sigma}(x_4),T(x_5),\ldots,T^\lambda(x_l),\ldots,T(x_n)) \\
& +P_9^{\mu\nu\rho\lambda}(x_1,\ldots,x_n)
\end{aligned}
\tag{4.1.20}
$$

where we can assume that:

$$
\begin{aligned}
P_3^{\mu\nu}(X)=0,\quad P_5^{\mu\nu\rho}=0,\quad P_7^{\mu\nu\rho\sigma}=0,\quad |X|=1,\\
P_6^{\mu\nu\rho}(X)=0,\quad P_8^{\mu\nu\rho\sigma}=0,\quad |X|\leq 2,\\
P_9^{\mu\nu\rho\sigma}(X)=0,\quad |X|\leq 3
\end{aligned}
\tag{4.1.21}
$$

without losing generality.

From (3.2.7), we get the following symmetry properties:

$$P_1(x_1,\ldots,x_n)\quad \text{is symmetric in}\quad x_1,\ldots,x_n; \tag{4.1.22}$$

$$P_2^{\mu}(x_1,\ldots,x_n)\quad \text{is symmetric in}\quad x_2,\ldots,x_n; \tag{4.1.23}$$

$$P_3^{\mu\nu}(x_1,\ldots,x_n)\quad \text{is symmetric in}\quad x_3,\ldots,x_n; \tag{4.1.24}$$

$$P_4^{\mu\nu}(x_1,\ldots,x_n)\quad \text{is symmetric in}\quad x_2,\ldots,x_n; \tag{4.1.25}$$

$$P_5^{\mu\nu\rho}(x_1,\ldots,x_n)\quad \text{is symmetric in}\quad x_3,\ldots,x_n; \tag{4.1.26}$$

$$P_6^{\mu\nu\rho}(x_1,\ldots,x_n)\quad \text{is symmetric in}\quad x_4,\ldots,x_n; \tag{4.1.27}$$

$$P_7^{\mu\nu\rho\sigma}(x_1,\ldots,x_n)\quad \text{is symmetric in}\quad x_3,\ldots,x_n; \tag{4.1.28}$$

$$P_8^{\mu\nu\rho\sigma}(x_1,\ldots,x_n)\quad \text{is symmetric in}\quad x_4,\ldots,x_n; \tag{4.1.29}$$

$$P_9^{\mu\nu\rho\sigma}(x_1,\ldots,x_n)\quad \text{is symmetric in}\quad x_5,\ldots,x_n; \tag{4.1.30}$$

we also have:

$$P_3^{\mu\nu}(x_1,\ldots,x_n)\quad \text{is antisymmetric in}\quad (x_1,\mu),(x_2,\nu); \tag{4.1.31}$$

$$P_4^{\mu\nu}=-P_4^{\nu\mu}; \tag{4.1.32}$$

$$P_5^{\mu\nu\rho}=-P_5^{\nu\mu\rho}; \tag{4.1.33}$$

$$P_6^{\mu\nu\rho}(x_1,\ldots,x_n)\quad \text{is antisymmetric in}\quad (x_1,\mu),(x_2,\nu),(x_3,\rho); \tag{4.1.34}$$

$$P_7^{\mu\nu\rho\sigma}=-P_7^{\nu\mu\rho\sigma}=-P_7^{\mu\nu\sigma\rho}; \tag{4.1.35}$$

$$P_7^{\mu\nu\rho\sigma}(x_1, x_2, \dots, x_n) = P_7^{\rho\sigma\mu\nu}(x_2, x_1, \dots, x_n); \tag{4.1.36}$$

$$P_8^{\mu\nu\rho\sigma} = -P_8^{\nu\mu\rho\sigma}; \tag{4.1.37}$$

$$P_8^{\mu\nu\rho\sigma}(x_1, x_2, x_3, \dots, x_n) = -P_8^{\mu\nu\sigma\rho}(x_1, x_3, x_2, \dots, x_n); \tag{4.1.38}$$

$$P_9^{\mu\nu\rho\sigma}(x_1, \dots, x_n) \quad \text{is antisymmetric in} \quad (x_1, \mu), (x_2, \nu), (x_3, \rho), (x_4, \sigma). \tag{4.1.39}$$

Let us note that for $n = 2$ only the first five relations (4.1.12)-(4.1.16) have to be checked; this can be done by some long but straightforward computations.

4.2 The Generic Structure of the Anomalies

If we apply the operator d_Q to the anomalous relations (4.1.12)-(4.1.20) we easily obtain again consistency relations of the Wess-Zumino type:

$$d_Q P_1(x_1, \dots, (x_n) = i \sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} P_2^\mu(x_l, x_1, \dots, \hat{x}_l, \dots, x_n) \tag{4.2.1}$$

$$d_Q P_2^\mu(x_1), \dots, (x_n) = i \frac{\partial}{\partial x_1^\nu} P_4^{\mu\nu}(x_1, \dots, x_n) - i \sum_{l=2}^n \frac{\partial}{\partial x_l^\nu} P_3^{\mu\nu}(x_1, x_l, x_2, \dots, \hat{x}_l, \dots, x_n) \tag{4.2.2}$$

$$\begin{aligned} d_Q P_3^{\mu\nu}(x_1, \dots, x_n) = i \frac{\partial}{\partial x_1^\rho} P_5^{\mu\rho\nu}(x_1, \dots, x_n) - i \frac{\partial}{\partial x_2^\rho} P_5^{\nu\rho\mu}(x_2, x_1, x_3, \dots, x_n) \\ + i \sum_{l=3}^n \frac{\partial}{\partial x_l^\rho} P_4^{\mu\nu\rho}(x_1, x_2, x_l, x_3, \dots, \hat{x}_l, \dots, x_n) \end{aligned} \tag{4.2.3}$$

$$d_Q P_4^{\mu\nu}(x_1, x_n) = i \sum_{l=2}^n \frac{\partial}{\partial x_l^\rho} P_5^{\mu\nu\rho}(x_1, x_l, x_2, \dots, \hat{x}_l, \dots, x_n) \tag{4.2.4}$$

$$\begin{aligned} d_Q P_5^{\mu\nu\rho}(x_1, \dots, x_n) = i \frac{\partial}{\partial x_2^\sigma} P_7^{\mu\nu\rho\sigma}(x_1, \dots, x_n) \\ -i \sum_{l=3}^n \frac{\partial}{\partial x_l^\sigma} P_8^{\mu\nu\rho\sigma}(x_1, x_2, x_l, x_3, \dots, \hat{x}_l, \dots, x_n) \end{aligned} \tag{4.2.5}$$

$$\begin{aligned} d_Q P_6^{\mu\nu\rho}(x_1, \dots, x_n) = i \frac{\partial}{\partial x_1^\sigma} P_8^{\mu\sigma\nu\rho}(x_1, \dots, x_n) \\ -i \frac{\partial}{\partial x_2^\sigma} P_8^{\nu\sigma\mu\rho}(x_2, x_1, x_3, \dots, x_n) + i \frac{\partial}{\partial x_3^\sigma} P_8^{\rho\sigma\mu\nu}(x_3, x_2, x_1, x_4, \dots, x_n) \\ -i \sum_{l=4}^n \frac{\partial}{\partial x_l^\rho} P_9^{\mu\nu\rho\sigma}(x_1, x_2, x_3, x_l, x_4, \dots, \hat{x}_l, \dots, x_n) \end{aligned} \tag{4.2.6}$$

$$d_Q P_i^{\mu\nu\rho\sigma}(x_1,\dots,x_n) = 0, \quad i = 7,8,9. \tag{4.2.7}$$

After a long computation (using the symmetry properties, the ghost number restrictions, etc. and making some convenient finite renormalizations) one can determine the generic form of the anomalies. One starts from a generic form of the same type as in the case of QED for all anomalies $P_i^{\dots}, \quad i = 1,\dots,9$ and determines that:

$$P_i^{\dots} = 0, \quad i = 3,\dots,9 \tag{4.2.8}$$

and $P_i^{\dots} = 0, \quad i = 1,2$ can be chosen of the form:

$$P_1 = \delta(X) W_1(x_1), \quad P_2^\mu = \delta(X) W_2^\mu(x_1). \tag{4.2.9}$$

The gauge invariance condition reduces to:

$$d_Q W_1 = i\partial_\mu W_2^\mu. \tag{4.2.10}$$

We give now the generic form of the Wick polynomials $W_i, \quad i = 1,2.$ fulfilling these conditions. First we have:

$$\begin{aligned} W_2^\mu = c_{abcd} :u_a u_b \Phi_c \partial^\mu \Phi_d: + c_{abc} :u_a u_b \partial^\mu \Phi_c: \\ + c_{ab;AB} :u_a u_b \overline{\psi}_A \gamma^\mu \psi_B: + c'_{ab;AB} :u_a u_b \overline{\psi}_A \gamma^\mu \gamma_5 \psi_B: \end{aligned} \tag{4.2.11}$$

where:

$$\begin{aligned} c_{abcd} = -(c \leftrightarrow d) = -(a \leftrightarrow b), \quad c_{ab;AB} = -(a \leftrightarrow b), \quad c'_{ab;AB} = -(a \leftrightarrow b) \\ c_{abcd} = 0 \quad \text{iff} \quad m_a + m_b + m_c + m_d \geq 0 \\ c_{abc} = 0, \quad m_a + m_b + m_c \geq 0, \quad c_{ab;AB} = 0, \quad c'_{ab;AB} = 0, \quad \text{iff} \quad m_a + m_b \geq 0. \end{aligned} \tag{4.2.12}$$

Finally we have:

$$\begin{aligned} W_1 = -2c_{abcd} :u_a A_b^\rho \Phi_c \partial_\rho \Phi_d: -2c_{abc} :u_a A_b^\rho \partial_\rho \Phi_c: \\ -2c_{ab;AB} :u_a A_b^\rho \overline{\psi}_A \gamma_\rho \psi_B: -2c'_{ab;AB} :u_a A_b^\rho \overline{\psi}_A \gamma_\rho \gamma_5 \psi_B: \\ + d'_{abc} (:u_a \partial_\rho \Phi_b \partial^\rho \Phi_c: + m_b m_c :u_a A_b^\rho A_{c\rho}: -2m_b :u_a A_b^\rho \partial_\rho \Phi_c:) \\ + d_{abc} :u_a \Phi_b \Phi_c: + d_{abcd} :u_a \Phi_b \Phi_c \Phi_d: + d_{abcde} :u_a \Phi_b \Phi_c \Phi_d \Phi_e: \\ + d_{ab;AB} :u_a \overline{\psi}_A \psi_B: + d'_{ab;AB} :u_a \overline{\psi}_A \gamma_5 \psi_B: \\ + f_{abc} :u_a F_b^{\rho\sigma} F_{c\rho\sigma}: + f'_{abc} \varepsilon_{\mu\nu\rho\sigma} :u_a F_b^{\mu\nu} F_c^{\rho\sigma}: \end{aligned} \tag{4.2.13}$$

where:

$$\begin{aligned} d_{abc} = 0 \quad m_a + m_b + m_c > 0, \quad d_{abcd} = 0 \quad m_a + m_b + m_c + m_d > 0, \\ d_{abcde} = 0, \quad m_a + m_b + m_c + m_d + m_e > 0, \\ f_{abc} = 0, \quad f'_{abc} = 0, \quad d'_{abc} = 0, \qquad m_a > 0, \\ d_{ab;AB} = 0, \quad d'_{ab;AB} = 0, \quad m_a + m_b > 0. \end{aligned} \tag{4.2.14}$$

There are no obvious arguments for the elimination of these anomalies. We remark a very interesting fact: if all the Bosons are heavy, then there the expression of the anomalies simplifies considerably.

We close with another interesting remark. Let us define the following differential forms:

$$\mathcal{T}_p(X) \equiv \sum T(A^{k_1}(x_1), \ldots, A^{k_p}(x_p)) dx_{1;k_1} \wedge \ldots \wedge dx_{p;k_p} \tag{4.2.15}$$

where we have defined in general:

$$dx_L \equiv dx \equiv dx^0 \wedge \ldots \wedge dx^3, \quad dx_\mu \equiv i_{\partial^\mu} dx, \quad dx_{\rho\sigma} \equiv i_{\partial^\rho} i_{\partial^\sigma} dx. \tag{4.2.16}$$

It is a very interesting fact that the following relation is true:

$$d^\rho \wedge dx_i = \sum_j c_i^{j;\rho} dx_j \tag{4.2.17}$$

where the constants $c_i^{j;\rho}$ are exactly the same as those appearing in (4.1.7). Then it is easy to prove that the induction hypothesis can be compactly written as

$$d_Q \mathcal{T}_p(X) = i d\mathcal{T}_p(X), \quad p = 1, \ldots, n-1 \tag{4.2.18}$$

and the anomalous gauge identity in order n is:

$$d_Q \mathcal{T}_n(X) = i d\mathcal{T}_n(X) + \mathcal{P}_n(X); \tag{4.2.19}$$

here the anomaly $\mathcal{P}_n(X)$ has an expression of the type (4.2.15):

$$\mathcal{P}_p(X) \equiv \sum P^{k_1,\ldots,k_p}(X) dx_{1;k_1} \wedge \ldots \wedge dx_{p;k_p} \tag{4.2.20}$$

with the identifications:

$$\begin{gathered} P^{L,\ldots,L} = P_1, \quad P^{\mu,L,\ldots,L} = P_2^\mu, \quad P^{\mu,\nu,L,\ldots,L} = P_3^{\mu\nu}, \\ P^{\mu\nu,L,\ldots,L} = P_4^{\mu\nu}, \quad P^{\mu\nu,\rho,L,\ldots,L} = P_5^{\mu\nu\rho} P^{\mu,\nu,\rho,L,\ldots,L} = P_6^{\mu\nu\rho}, \\ P^{\mu\nu,\rho\sigma,L,\ldots,L} = P_7^{\mu\nu\rho\sigma}, \quad P^{\mu\nu,\rho,\sigma,L,\ldots,L} = P_8^{\mu\nu\rho\sigma}, \quad P^{\mu,\nu,\rho,\sigma,L,\ldots,L} = P_9^{\mu\nu\rho\sigma}. \end{gathered} \tag{4.2.21}$$

So, the expressions $\mathcal{P}_p(X)$ are differential forms with coefficients quasi-local operators. Let us denote by $\mathcal{A}$ this class of differential forms. From (4.2.19) we easily obtain the consistency equation

$$d_Q \mathcal{P}_n(X) + i d\mathcal{P}_n(X) = 0 \tag{4.2.22}$$

which is the compact form of the relations (4.2.1) - (4.2.7). One can "solve" this equation using the homotopy operator p of the de Rham complex: we have

$$\mathcal{P}_n(X) = d(p\mathcal{P}_n(X)) + d_Q(ip\mathcal{P}_n(X)). \tag{4.2.23}$$

It is tempting to argue that by the finite renormalization

$$\mathcal{T}_n(X) \rightarrow \mathcal{T}_n(X) + ip\mathcal{P}_n(X) \tag{4.2.24}$$

the anomalies are eliminated. However one can check that if we apply the homotopy operator p on a element from $\mathcal{A}$ we do not obtain a element from $\mathcal{A}$. It follows that the finite renormalization given above is not legitimate and the argument has to be modified somehow. However, let us notice the interesting fact that the usual expression of the homotopy operator for the de Rham complex is constructed using the action of the dilation group. This is in agreement to the role played by this group in the traditional approach to the non-renormalizability theorems.

References

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